Solutions | Sheet 1

Q1 a) In general: $\langle E \rangle = \sum_{k=0}^{\infty} \epsilon_k f_k = \frac{1}{Z} \sum_{k=0}^{\infty} d_k \epsilon_k e^{-\frac{\epsilon_k}{k_B T}}$

For EM-waves: 2 Polarizations $\Rightarrow d_k = 2$

$$\Rightarrow \langle E \rangle = \frac{1}{Z} \sum_{k=0}^{\infty} 2h\nu k \cdot e^{-\frac{h\nu k}{k_B T}}$$

1.) $$Z = 2 \sum_{k=0}^{\infty} e^{-\frac{h\nu k}{k_B T}} = 2 \cdot \sum_{k=0}^{\infty} \left[e^{-\frac{h\nu}{k_B T}} \right]^k$$

$$= 2 \cdot \frac{1}{1 - e^{-\frac{h\nu}{k_B T}}}$$

2.) $$\sum_{k=0}^{\infty} h\nu k \, e^{-\frac{h\nu k}{k_B T}} = - \frac{\partial}{\partial \left(\frac{1}{k_B T} \right)} \sum_{k=0}^{\infty} e^{-\frac{h\nu k}{k_B T}} .$$

$$= - \frac{\partial}{\partial \left(\frac{1}{k_B T} \right)} \frac{1}{1 - e^{-\frac{h\nu}{k_B T}}}$$

$$= - \frac{-1}{\left(1 - e^{-\frac{h\nu}{k_B T}} \right)^2} \cdot \left(+ h\nu \, e^{-\frac{h\nu}{k_B T}} \right)$$

$$= \frac{h\nu e^{-\frac{h\nu}{k_B T}}}{\left(1 - e^{-\frac{h\nu}{k_B T}} \right)^2}$$

$$\Rightarrow \langle E \rangle = \frac{\cancel{x} \frac{h\nu\, e^{\frac{h\nu}{k_BT}}}{(1-e^{-h\nu/k_BT})^{\cancel{2}}}}{\cancel{x} \frac{1}{\cancel{1-e^{-h\nu/k_BT}}}} = \frac{h\nu\, e^{-\frac{h\nu}{k_BT}}}{1-e^{-\frac{h\nu}{k_BT}}} = \frac{h\nu}{e^{\frac{h\nu}{k_BT}}-1}$$

Q1 b) The mode density of space is

$$\tilde{M}(\nu) = \frac{8\pi\nu^2}{c^3}$$

Thus the energy density will be given by

$$u(\nu) = \tilde{M}(\nu) \cdot \langle E \rangle$$

$$= \frac{8\pi\nu^2}{c^3} h\nu \frac{1}{e^{\frac{h\nu}{k_BT}}-1}$$

$$= \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{\frac{h\nu}{k_BT}}-1}$$

q.e.d.

Q1 c) The same argument results in

$$u(\nu) = \tilde{M}(\nu) \cdot E$$

$$= \frac{8\pi\nu^2}{c^3} k_BT$$

q.e.d.

P2 a)

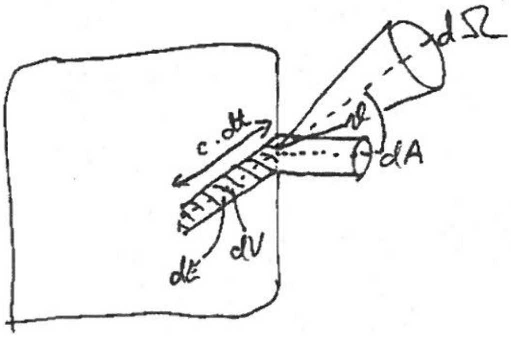
$$\frac{E}{V} = \int_0^\infty u(\nu)\,d\nu = \int_0^\infty \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{\frac{h\nu}{k_BT}}-1}\,d\nu$$

$$= \frac{8\pi h}{c^3}\int_0^\infty \frac{\nu^3}{e^{\frac{h\nu}{k_BT}}-1}\,d\nu \qquad x = \frac{h\nu}{k_BT} \qquad d\nu = \frac{k_BT}{h}dx$$

$$= \frac{8\pi h}{c^3}\left(\frac{k_BT}{h}\right)^4 \int_0^\infty \frac{x^3}{e^x-1}\,dx$$

$$= \frac{8\pi^5 k_B^4}{15 h^3c^3} T^4$$

b)



The emitted energy into the solid angle $d\Omega$ is

$$dE = \frac{E}{V}\cdot \overbrace{dA\cos\vartheta\cdot c\,dt}^{dV}\cdot\frac{d\Omega}{4\pi}$$

$$d\Omega = \sin\vartheta\, d\vartheta\, d\varphi$$

The total radiated intensity thus is: $dI = \frac{dP}{dA} = \frac{dE}{dA\,dt} = \frac{E}{V}\cdot c\cos\vartheta\,\frac{d\Omega}{4\pi}$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}}\sin\vartheta\,d\vartheta\int_0^{4\pi}d\varphi\,\frac{E}{V}\,c\cos\vartheta\cdot\frac{1}{4\pi}$$

$$= \frac{c}{4\pi}\frac{E}{V}\cdot 2\pi\cdot\int_0^{\frac{\pi}{2}}\sin\vartheta\cos\vartheta\,d\vartheta$$

$$= \frac{c}{2}\frac{E}{V}\cdot\frac{1}{2} = \frac{c}{4}\frac{E}{V}$$

$$= \frac{2\pi^5 k_B^4}{15h^3c^2}T^4 \quad \Rightarrow \quad \text{c)} \quad \boxed{\sigma = \frac{2\pi^5 k_B^4}{15h^3c^2} = 5.67\cdot 10^{-8}\frac{W}{m^2K^4}}$$

3 a) $$u(\lambda) = \frac{8\pi hc}{\lambda^5} \frac{1}{e^{\frac{hc}{\lambda k_B T}} - 1}$$

Max. $\Rightarrow$ $\frac{\partial u}{\partial \lambda} = 0$

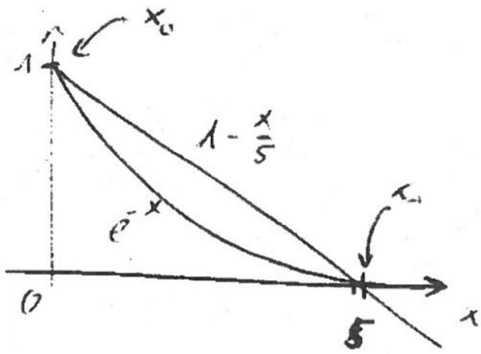
$$\Leftrightarrow -\frac{8\pi hc}{\lambda^6} \frac{5}{e^{\frac{hc}{\lambda k_B T}} - 1} + \frac{8\pi hc}{\lambda^5} \cdot \frac{-1}{\left(e^{\frac{hc}{\lambda k_B T}} - 1\right)^2} \left(-\frac{hc}{\lambda^2 k_B T}\right) e^{\frac{hc}{\lambda k_B T}} \stackrel{!}{=} 0$$

$$\Leftrightarrow \frac{8\pi hc}{\lambda^6} \frac{1}{e^{\frac{hc}{\lambda k_B T}} - 1} \left[5 - \frac{hc}{\lambda k_B T} \cdot \frac{e^{\frac{hc}{\lambda k_B T}}}{e^{\frac{hc}{\lambda k_B T}} - 1} \right] \stackrel{!}{=} 0$$

~~with~~ with $x = \frac{hc}{\lambda k_B T}$

$$\Leftrightarrow \quad 5 = x \frac{e^x}{e^x - 1} \quad \Leftrightarrow \quad 5(1 - e^{-x}) = x$$

$$\Leftrightarrow \quad e^{-x} = 1 - \frac{x}{5}$$



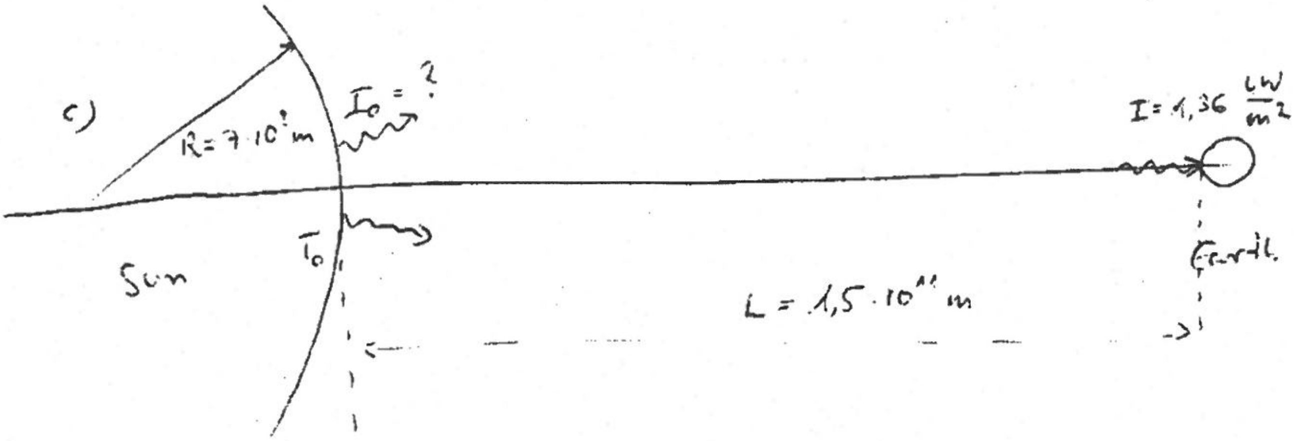
$\Rightarrow \quad x_0 = 0 \quad \vee \quad x_1 \approx 4{,}9651\ldots$

$$\Rightarrow \quad \frac{hc}{\lambda k_B T} = x_1 \quad \Rightarrow \quad \lambda_{max} \cdot T = \frac{hc}{k_B x_1} = \text{const.}$$

P3 b)

$$x = \frac{hc}{k_B x_1} \approx 2.9\ \text{mm K}$$

P3 c)



$$I \propto \frac{1}{r^2} \Rightarrow \frac{I_0}{I} = \frac{(L+R)^2}{R^2} \Rightarrow I_0 = I \cdot \frac{(L+R)^2}{R^2} = 46348 \times I$$

$$= 63{,}0\ \frac{MW}{m^2}\ !!$$

$$I_0 = \sigma T_0^4 \Rightarrow T_0 = \sqrt[4]{\frac{I_0}{\sigma}} = \underline{\underline{5774\ K}}$$

$$\Rightarrow \lambda_{max} = \frac{2.9\ \text{mm}}{T_0} = \underline{\underline{502{,}3\ \text{nm}}}$$